NONLINEAR PDE WITH ROUGH (STOCHASTIC) TIME DEPENDENCE: APPLICATIONS AND THEORY

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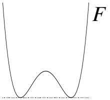
- plan of the talk
  - **applications**
    - phase transitions and front propagation
    - (stochastic) selection principle
    - pathwise vs in-the-mean stochastic control theory
  - **pde with multiplicative rough time dependence**
    - difficulties (conceptual and technical)
    - well-posedness and qualitative properties
  - **some qualitative properties**
    - domain of dependence and speed of propagation
    - intermittent regularity
    - long time behavior
    - regularity of solutions
  - **concluding remarks and open problems**

- phase transitions and front propagation

microscopic (IPS) and macroscopic (phase field theory) models $\Rightarrow$

reaction diffusion equations perturbed by additive noise

$$u_t - \Delta u = F'(u) + \dot{w}_{x,t}$$



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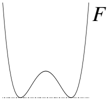

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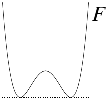

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level set stochastic pde

$$du = \mathrm{tr}[(I - \frac{Du \otimes Du}{|Du|^2})D^2u]\, dt + |Du| \cdot \dot{w}_{x,t}$$

- the KOJ-claim

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- conjecture is false

  - perturbation **“too violent”** to preserve the stability properties of minima of F

  - motion of interfaces with $V = \dot{w}_{x,t}$ is **not** well defined

    “solutions” of $u_t = |u_x|\dot{w}_{x,t}$ **blow up**

    multiplicative white noise dependence is too strong!

interface moving with normal velocity $V = \dot{w}_{x,t}$

$\dot{w}_{x,t} = \sum_{k=0}^{\infty} \phi_k(x) dB_t^k$ dB^k time white noise $(\phi_k)_k$ basis of L_x^2

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level set pde $du = |u_x| \cdot \dot{w}_{x,t}$ $u(\cdot, 0) = u_0$ **is not well-posed**

well-posedness and u_0 increasing $\Rightarrow$ solution increasing

$du = u_x \dot{w}_{x,t}$ $u(\cdot, 0) = u_0$

solution is given by characteristics $dX_t = \sum_{k=0}^{\infty} \phi_k(X_t) dB_t^k$

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approximate $dX_t^N = \sum_{k=0}^{N} \phi_k(X_t) dB_t^k$

$dX_t^N = \sum_{k=0}^{N} \cos(kX_t^N) dB_t^k + \sum_{k=0}^{N} \sin(kX_t^N) dB_t^k$

$\mathbb{E}[X_t^N] = x$ and $\frac{d}{dt} \mathbb{E}|X_t^N|^2 \approx N + 1$

no convergence

- $d = 1$ $\quad du = (u_{xx} + F'(u))dt + a(x)\dot{w}_{x,t}$ $\quad a$ compactly supported

  appropriate scaling yields interface $(x_t)_{t \geq 0}$ evolving by

  $dx = b(x)dt + \sigma(x)dB$ $\quad B$ Brownian motion in time Funaki

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- $d > 1$ dB white noise only in time

  "correct scalling" $\Rightarrow$ $du^\varepsilon = \Delta u^\varepsilon dt + \frac{1}{\varepsilon}(F'(u^\varepsilon)dt + \varepsilon^{1/2}dB)$

  problem still not ok dB is **still** too strong

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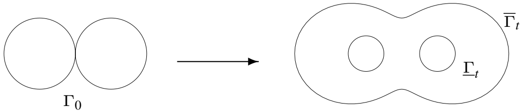
  replace dB by " **mild approximation**" $\dot{B}^\varepsilon$

  $u_t^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon}(F'(u^\varepsilon) + \varepsilon^{1/2}\dot{B}^\varepsilon)$ $\qquad \Rightarrow \qquad$ $V = -\mathrm{tr}Dn + dB$

  Funaki $\quad d = 2$ interface smooth
  Yip $\quad$ variational method with convex fronts
  Lions and S. $\quad d \geq 2$ global in time — no smoothness

- a stochastic selection principle

" unstable configurations" for motion by mean curvature



is there a stochastic mechanism that selects at the limit a unique interface?

- a stochastic selection principle

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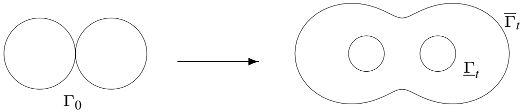

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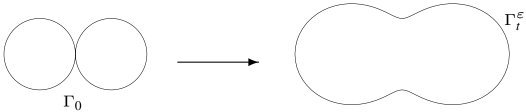
$$V_n = -\text{mean curvature} + \varepsilon dB$$

converges a.s. to the maximal solution of the motion without the noise



Dirr, Luckhaus and Novaga — short time smooth fronts

Yip and S. — global in time, no regularity

- pathwise stochastic control theory

“controls” constrained with respect to measurability legality in math finance

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W, B independent Brownian motions with filtrations $(\mathcal{F}_t^W)_{t\geq 0}, (\mathcal{F}_t^B)_{t\geq 0}$

$\mathcal{A}$ the set of $\mathcal{F}_t^W$-progressively measurable controls $(\pi_t)_{t\geq 0}$

dynamics

uncontrolled process $dY = h(Y)dt + \sqrt{2}\sigma_1(Y)dW + \sigma_2(Y) \circ dB \quad Y_t = y$

controlled process $dX = \pi(\mu dt + \sqrt{2}\sigma(Y))dW \quad X_t = x$

value function $u(x, y, t) = \text{esssup}_{\pi \in \mathcal{A}} \mathbb{E}_{x,t} \left[g(X_T) + \int_t^T l(X_s, Y_s) ds \,\middle|\, \mathcal{F}_T^B \right]$

system is optimized in X given the additional information coming from Y

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pathwise Bellman equation

$$du = \left(\sup_{\pi\in A}\left[\pi^2\sigma^2 u_{xx} + \pi\mu u_x\right] + \left[\sigma_1^2 u_{yy} + hu_y + l(x,y)\right]\right)dt + \sigma_2 u_y \circ dB$$

- equations with multiplicative “rough” time - pathwise/stochastic solutions

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$$du = \sum_{i=1}^m H^i(Du,x)\cdot d\omega^i + F(D^2u, Du, u, x)dt \quad \text{in} \quad \mathbb{R}^d \times (0,\infty)$$

$$du = \sum_{i=1}^d A^i(u)_{x_i}\cdot d\omega^i + \operatorname{div}\left[A(u)Du\right] dt \quad \text{in} \quad \mathbb{R}^d \times (0,\infty)$$

  - $\omega = (\omega^1, \ldots, \omega^m)$ **continuous** (Brownian or rough path)

  enough to be able to solve the characteristics

$$dX = -\sum_{i=1}^m D_x H^i(P,X)\cdot d\omega^i \qquad dP = \sum_{i=1}^m D_p H^i(P,X)\cdot d\omega^i$$

  - $u \in \mathbb{R}$ $\quad F$ degenerate elliptic
  - F and H may depend on t
  - ω regular in x KPZ is outside the scope of the theory

to simplify consider

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- **"deterministic" viscosity solutions "nice" time dependence**
  - in general shocks (discontinuities of Du) appear in finite time
  - viscosity solutions $\omega \in \mathrm{BV} \Rightarrow \exists!$ solution $u \in C_{x,t}$ and comparison

$$\|(u-v)_\pm(\cdot, t)\| \leq \|(u_0 - v_0)_\pm\|$$

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- **is it possible to extend theory to** $\omega \in C$?

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    $u_t = |u_x|\dot{\omega} \quad u(x, 0) = |x|$

    $$u(x,t) = \max\left((x + \omega(t))_+, \max_{0 \le s \le t} \omega(s)_+\right)$$

“THEOREM” $\exists!$ a well-posed solution for any continuous paths

IFF

H is the difference of two convex functions

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- **solutions are continuous in H and ω**
- **solutions of problems with regularized H and ω converge to the same limit**

  $u_{\varepsilon,t} = H_\varepsilon(Du_\varepsilon)\dot{\omega}_\varepsilon$ $H_\varepsilon, \omega_\varepsilon$ smooth approxs of H and ω $\Rightarrow$ $\|u_\varepsilon - u\| \underset{\varepsilon\to 0}{\to} 0$

- **the contraction property**

  $\|(u-v)_\pm(\cdot,t)\| \leq \|(u_0 - v_0)_\pm\|$

- **more regularity on ω requires less regularity on H**
- **theory extends to** $du = F(D^2u, Du, x)dt + \sum_{i=1}^m H^i(Du,x)\cdot dB^i$
- **convergence of monotone finite-difference in space-time approximations**
- **qualitative properties** (next)

- domain of dependence - finite speed of propagation Gassiat, Gess, Lions, S.

$$u_1(\cdot,0) = u_2(\cdot,0) \text{ in } B(0,R_0) \quad \Rightarrow \quad u_1(\cdot,t) = u_2(\cdot,t) \text{ in } B(0,R(t))?$$

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  - **a positive result**

    H difference of two convex functions

    $u(\cdot,0) \equiv A \text{ in } B(0,R) \quad \Rightarrow \quad u(\cdot,t) \equiv A \text{ in } B(0,R(t))$

    $R(t) = R - L(\max_{s\in[0,T]}\omega(s) - \min_{s\in[0,T]}\omega(s))$

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  - **a negative result**   Gassiat

    $$u_t = (|u_x| - |u_y|)\dot\omega$$

    $$u(x,y,0) = |x-y| + \Theta(x,y) \qquad \begin{cases}\Theta \geq 0\\ \Theta(x,y) \geq 1 \text{ if } x,y \geq R\end{cases}$$

    $u(0,0,T) > 0$ if $\|\omega\|_{\mathrm{TV}_{[0,T]}} > R$

    ω Brownian motion $\|\omega\|_{\mathrm{TV}_{[0,T]}} = \infty$ no domain of dependence

- finite speed of propagation

$$du = H(Du, x) \cdot d\omega \quad H \text{ convex in } p \quad \omega \in C_0([0, T])$$

$$\rho_H(\omega, T) = \sup \left\{ R \geq 0 : \; u^1(\cdot, 0) = u^2(\cdot, 0) \text{ in } B_R(0) \; u^1(0, T) \neq u^2(0, T) \right\}$$

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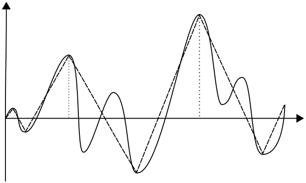
skeleton $R_{0,T}(\omega)$



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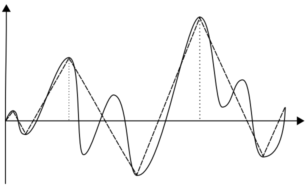
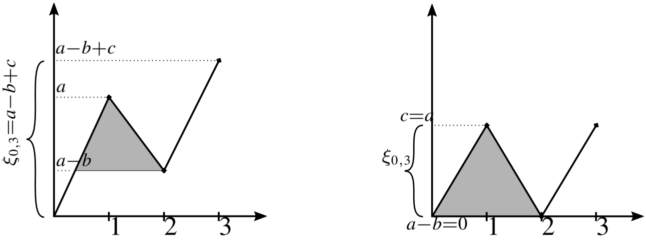


- **results**
  - $u^{\omega}(\cdot, T) = u^{R_{0,T}(\omega)}(\cdot, T) \quad \Rightarrow \quad \rho_H(\omega, T) \leq L \, \|R_{0,T}(\omega)\|_{TV([0,T])}$
  - B Brownian motion $\Rightarrow$ $\|R_{0,T}(B)\|_{TV([0,T])} < \infty$ a.s.
  - $H(p) = |p| \;\; \Rightarrow \;\; \rho_H(\omega, T) \geq \|R_{0,T}(\omega)\|_{TV([0,T])}$

main step of the proof – **control of cancellations**

- $S_H(c) \circ S_{-H}(b) \circ S_H(a) = S_H(a+c-b)$ for $a, c > b$



- intermittent regularity

$$du = H(Du) \cdot d\omega \qquad \omega \in C_0([0,\infty))$$

- think of solution as the outcome of "**repeated up and downs**"
- oscillations and cancellations may yield "regularization"
- revisit the regularizing and propagation of regularity properties of

  $u_t = \pm H(Du)$ in $\mathbb{R}^d \times (0,\infty)$ and H uniformly convex

  to obtain estimates which **can be iterated in time**

- intermittent regularity

$u_t = H(Du) \cdot d\omega \qquad \omega \in C_0([0,\infty)) \qquad \theta I \le D^2 H \le \Theta I$

running max $M(t) = \max_{0 \le s \le t} \omega(s)$

running min $m(t) = \min_{0 \le s \le t} \omega(s)$

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- $$\|Du(\cdot,t)\| \le \sqrt{\frac{2\|u(\cdot,t)\|}{\theta(M(t)-m(t))}}$$

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- $d = 1$

$$\frac{1}{\omega(t)-m(t)} \le H''(u_x(\cdot,t))u_{xx}(\cdot,t) \le \frac{1}{M(t)-\omega(t)}$$
$$\Downarrow$$
$$\|u_{xx}(\cdot,t)\| \le C\max\{\frac{1}{\omega(t)-m(t)}, \frac{1}{M(t)-\omega(t)}\}$$

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$$\|u_{xx}(\cdot,t)\| \le C \max\{\frac{1}{\omega(t)-m(t)}, \frac{1}{M(t)-\omega(t)}\}$$

- estimate true when $d > 1$ iff $H = |p|^2$

- intermittent regularity

$u_t = H(Du) \cdot d\omega \qquad \omega \in C_0([0,\infty)) \qquad \theta I \leq D^2 H \leq \Theta I$

running max $M(t) = \max_{0 \leq s \leq t} \omega(s)$

running min $m(t) = \min_{0 \leq s \leq t} \omega(s)$

- $$\|Du(\cdot,t)\| \leq \sqrt{\frac{2\|u(\cdot,t)\|}{\theta(M(t)-m(t))}}$$
- $d = 1$

$$\frac{1}{\omega(t)-m(t)} \leq H''(u_x(\cdot,t))u_{xx}(\cdot,t) \leq \frac{1}{M(t)-\omega(t)}$$

$$\Downarrow$$

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- estimate true when $d > 1$ iff $H = |p|^2$
- ω Brownian motion $\quad \Rightarrow$

  $u(\cdot,t) \in C^{0,1}(\mathbb{R}^d)$ and, when $d = 1$, $u(\cdot,t) \in C^{1,1}(\mathbb{R})$ off an uncountable subset of $(0,\infty)$ with no isolated points and Hausdorff dimension $1/2$

- "deterministic" problem with convex Hamiltonians revisited

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- **bounds cannot be iterated**

- **sharp** “deterministic” bounds

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$W(Du) = F(Du)D^2uF(Du)$ **solves** $W_t = DHDW + |W|^2$

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- "mission" **almost** accomplished
  - lower bound is **always** true
  - upper bound **requires** $u(\cdot,0) \in C^{1,1}$ **unless** $H(p) = (Ap,p)$
    A independent of p

- long time behavior

$$\begin{cases} du = H(Du)\cdot d\omega \\ u(\cdot,0) = u_0 \end{cases}$$

$$\begin{cases} H \text{ continuous}, \ u_0 \ \text{continuous and ``periodic'' in } x \\ \omega \ \text{ continuous and } \ \omega(0) = 0 \\ H(0) = 0 \quad \Rightarrow \ \text{constants are solutions} \end{cases}$$

$u(\cdot,t) \underset{t\to\infty}{\to} u^\infty$ **with** u^∞ **constant in** x **and depending only on** u_0 **and** ω **?**

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  - $H(p) = v \cdot p \quad v \in \mathbb{R}^d$

    $u(x,t) = u_0(x + v\omega(t))$ no convergence
  - $H(p) = |p|^2 \quad \dot{\omega}(t) \geq 0$ and $\omega(t) \underset{t\to\infty}{\to} \infty$

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- **what happens if** ω **oscillates ?**

- behavior as $t \to \infty$

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$d = 1 \quad \omega$ Brownian motion $\quad u_0$ periodic

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$$d(\textstyle\int_0^1 u dx) = (\int_0^1 H(u_x) dx) dB \quad \Rightarrow \qquad (\int_0^1 \phi(u_x) u_{xx} = 0)$$

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what if H is more regular, for example, convex?

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- $u_t^\varepsilon = H(u_x^\varepsilon)\dot{B}^\varepsilon \Rightarrow \frac{d}{dt}\int_0^1 u^\varepsilon(x,t)dx = \int_0^1 H(u_x(x,t))dx\dot{B}^\varepsilon \Rightarrow$

  $\int_0^1 u^\varepsilon(x,t)dx - \int_0^1 u^\varepsilon(x,s)dx = \int_s^t \int_0^1 H(u_x^\varepsilon(x,\tau))dxdB^\varepsilon(\tau)$

  $\underset{\varepsilon\to 0}{\Rightarrow} \int_0^1 u(x,t)dx - \int_0^1 u(x,s)dx = \int_s^t \int_0^1 H(u_x(x,\tau))dxdB(\tau) \Rightarrow$

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- (some) open problems
  - **multiple paths** — nontrivial interaction among the different paths and cancellations
  - **stochastic behavior of solutions** — theory so far has been mainly pathwise
  - **higher regularity**