Mean Field Limits for Coulomb-type Flows

Sylvia Serfaty

Courant Institute, NYU

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The discrete minimization problem

- Consider the energy

$$H_N(x_1,\dots,x_N)=\frac{1}{2}\sum_{1\le i\neq j\le N} w(x_i-x_j)+N\sum_{i=1}^N V(x_i)\quad x_i\in\mathbb{R}^d$$

- interaction potential

$$w(x)=-\log|x|\quad d=1,2\quad \textbf{log case}$$

$$\text{or}\quad w(x)=\frac{1}{|x|^{d-2}}\quad d\ge 3\quad \textbf{Coulomb case}$$

more generally

$$w(x)=\frac{1}{|x|^s}\quad \textbf{Riesz case}$$

- V confining potential, sufficiently smooth and growing at infinity
- Motivations: Fekete points (approximation theory), vortices, dislocations, statistical mechanics...

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Minimizers and equilibrium measure

For minimizers we have

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \to \mu_V \qquad \frac{1}{N^2}\min H_N \to \mathcal{E}(\mu_V)$$

where μ_V = **Frostman equilibrium measure** is the unique minimizer of

$$\mathcal{E}(\mu) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} w(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^d} V(x)\,d\mu(x).$$

Makes sense only if w integrable near $0 \Leftrightarrow s < d$
Exists only if V grows fast enough at ∞.

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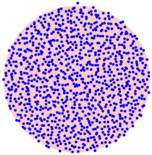
Examples

- $V(x) = |x|^2$, Coulomb case, then $\mu_V = \frac{1}{c_d}\mathbf{1}_{B_1}$ (**circle law**).



- $d = 1$, $w = -\log|x|$, $V(x) = x^2$ then $\mu_V(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{|x|<2}$ (**semi-circle law**)

- we will always assume μ_V is a measure with a (regular) density, and identify the measure and its density

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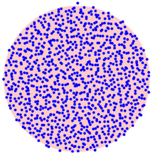

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$$\boxed{\exists c \text{ s.t. } \quad h^{\mu_V} + V = c \text{ in the support of } \mu_V}$$

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The discrete time-dependent problem

Consider

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Evolution equation

$$\dot{x}_i = -\frac{1}{N}\nabla_i H_N(x_1,\dots,x_N) \qquad \textbf{gradient flow}$$

$$\dot{x}_i = -\frac{1}{N}\mathbb{J}\nabla_i H_N(x_1,\dots,x_N) \qquad \textbf{conservative flow} \quad (\mathbb{J}^T = -\mathbb{J})$$

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Formal limit

Consider the *empirical measure*

$$\mu_N^t := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i^t}$$

We formally expect $\mu_N^t \rightharpoonup \mu^t$ where μ^t solves

$$\boxed{\partial_t \mu = \mathrm{div}\,\left(\nabla(w*\mu)\mu\right)} \qquad (MFD)$$

in the dissipative case or

$$\boxed{\partial_t \mu = \mathrm{div}\,\left(\mathbb{J}\nabla(w*\mu)\mu\right)} \qquad (MFC)$$

in the conservative case.

Such a result is equivalent to *propagation of molecular chaos*: if $f_N^0(x_1,\dots,x_N) = \mu^0(x_1)\dots\mu^0(x_N)$ is the density of probability of having initial positions at $(x_1,\dots,x_N)$ then $f_N^t \rightharpoonup \mu^t(x_1)\dots\mu^t(x_N)$.

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Previous results

- [Schochet '96, Goodman-Hou-Lowengrub '90] ($d = 2$ log) (point vortex system)
- [Hauray' 09, Carrillo-Choi-Hauray '14] ($s < d - 2$) stability in Wasserstein W_∞
- [Carrillo-Ferreira-Precioso '12, Berman-Onnheim '15] ($d = 1$) Wasserstein gradient flow, use *convexity* of the interaction in 1D
- [Duerinckx '15] ($d \leq 2$, $s < 1$) modulated energy method
- for convergence to Vlasov-Poisson [Hauray-Jabin '15, Jabin-Wang '17] $s < d - 2$. Coulomb interaction (or more singular) remains open.

The modulated energy method

Idea: use Coulomb (or Riesz) based metric:

$$\|\mu - \nu\|^2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} w(x-y) d(\mu-\nu)(x) d(\mu-\nu)(y).$$

Observe weak-strong uniqueness property of the solutions to (MFD)-(MFC) for $\|\cdot\|$:

$$\|\mu_1^t - \mu_2^t\|^2 \leq e^{Ct}\|\mu_1^0 - \mu_2^0\|^2 \qquad C = C(\|\nabla^2(w * \mu_2)\|_{L^\infty})$$

In the discrete case, let X_N denote $(x_1, \ldots, x_N)$ and take for modulated energy,

$$F_N(X_N^t, \mu^t) = \frac{1}{N^2} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} w(x-y) d\Big(\sum_{i=1}^N \delta_{x_i^t} - N\mu^t\Big)(x) d\Big(\sum_{i=1}^N \delta_{x_i^t} - N\mu^t\Big)(y)$$

where $\triangle$ denotes the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, and μ^t solves (MFD) or (MFC).

Analogy with "relative entropy" and "modulated entropy" methods [Dafermos '79] [DiPerna '79] [Yau '91] [Brenier '00]...

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Theorem (S. '18)

Assume (MFD) resp. (MFC) admits a solution

$$\begin{cases} \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d)), & \text{if } s < d-1 \\ \mu^t \in L^\infty([0,T], C^\sigma(\mathbb{R}^d)) \text{ with } \sigma > s-d+1, & \text{if } s \geq d-1. \end{cases}$$

with $\nabla^2 w * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$. *There exist constants* C_1, C_2 *depending on the norms of* μ^t *and* $\beta > 0$ *depending on* d, s, σ, *s.t.* $\forall t \in [0,T]$

$$F_N(X_N^t, \mu^t) \leq \left(F_N(X_N^0, \mu^0) + C_1 N^{-\beta}\right) e^{C_2 t}.$$

In particular, if $\mu_N^0 \rightharpoonup \mu^0$ *and is such that*

$$(*) \qquad \lim_{N\to\infty} F_N(X_N^0, \mu^0) = 0,$$

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Comments on the assumptions

- well-prepared assumption $(*)$ implied by

$$\lim \frac{1}{N^2} H_N(X_N^0) = \iint w(x-y) d\mu^0(x) d\mu^0(y).$$

- regularity assumption on μ^t allow for "patches" i.e. measures which are only L^∞, as in vortex patch solutions to Euler's eq [Chemin, Serfati]
- Self-similar solutions of patch type are attractors in the Coulomb case ([S-Vazquez]). For general s, self-similar *Barenblatt solutions* of the form

$$t^{-\frac{d}{2+s}} \left(a - bx^2 t^{-\frac{2}{2+s}}\right)_+^{\frac{s-d+2}{2}}$$

- limiting equation called fractional porous medium equation
- required propagation of regularity ok for $s < d-1$ ([Lin-Zhang, Xiao-Zhou, Caffarelli-Vazquez, Caffarelli-Soria-Vazquez,] open for $s > d-1$

Proof of the weak-strong uniqueness principle

Set $h^\mu = w * \mu$. In the Coulomb case

$$-\Delta h^\mu = c_d \mu$$

We have by IBP

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} w(x-y) d\mu(x) d\mu(y) = \int_{\mathbb{R}^d} h^\mu d\mu = -\frac{1}{c_d} \int_{\mathbb{R}^d} h^\mu \Delta h^\mu = \frac{1}{c_d} \int_{\mathbb{R}^d} |\nabla h^\mu|^2.$$

Stress-energy tensor

$$[\nabla h^\mu]_{ij} = 2\partial_i h^\mu \partial_j h^\mu - |\nabla h^\mu|^2 \delta_{ij}.$$

For regular μ,

$$\mathrm{div}\ [\nabla h^\mu] = 2\Delta h^\mu \nabla h^\mu = -\frac{2}{c_d} \mu \nabla h^\mu.$$

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Let μ_1 and μ_2 be two solutions to (MFD) and $h_i = w * \mu_i$.

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} |\nabla(h_1 - h_2)|^2 &= 2c_d \int_{\mathbb{R}^d} (h_1 - h_2)\partial_t(\mu_1 - \mu_2) \\
&= 2c_d \int_{\mathbb{R}^d} (h_1 - h_2) \mathrm{div}\, (\mu_1 \nabla h_1 - \mu_2 \nabla h_2) \\
&= -2c_d \int_{\mathbb{R}^d} (\nabla h_1 - \nabla h_2) \cdot (\mu_1 \nabla h_1 - \mu_2 \nabla h_2) \\
&= -2c_d \int_{\mathbb{R}^d} |\nabla(h_1 - h_2)|^2 \mu_1 - 2c_d \int_{\mathbb{R}^d} \nabla h_2 \cdot \nabla(h_1 - h_2)(\mu_1 - \mu_2) \\
&\le -2c_d \int_{\mathbb{R}^d} \nabla h_2 \cdot \mathrm{div}\, [\nabla(h_1 - h_2)]
\end{aligned}$$

so if $\nabla^2 h_2$ is bounded, we may IBP and bound by

$$\|\nabla^2 h_2\|_{L^\infty} \int_{\mathbb{R}^d} |[\nabla(h_1 - h_2)]| \le 2\|\nabla^2 h_2\|_{L^\infty} \int_{\mathbb{R}^d} |\nabla(h_1 - h_2)|^2,$$

$\rightsquigarrow$ result by Gronwall's lemma. In discrete case, control instead

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} (\nabla h^{\mu}(x) - \nabla h^{\mu_N}(y)) \cdot \nabla w(x-y) d(\mu - \mu_N)(x) d(\mu - \mu_N)(y)$$

Use suitable *truncations* of the potentials $w * (\sum_i \delta_{x_i} - N\mu)$

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The Ginzburg-Landau equations

$u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$

$$-\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2)$$ Ginzburg-Landau equation (GL)

$$\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)$$ parabolic GL equation (PGL)

$$i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)$$ Gross-Pitaevskii equation (GP)

Associated energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}$$

Models: superconductivity, superfluidity, Bose-Einstein condensates, nonlinear optics

Vortices

- in general $|u| \leq 1$, $|u| \simeq 1$ = superconducting/superfluid phase, $|u| \simeq 0$ = normal phase
- u has zeroes with nonzero degrees = **vortices**
- $u = \rho e^{i\varphi}$, characteristic length scale of $\{\rho < 1\}$ is ε = vortex core size
- degree of the vortex at x_0:

$$\frac{1}{2\pi} \int_{\partial B(x_0, r)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

- In the limit $\varepsilon \to 0$ vortices become *points*, (or curves in dimension 3).

Vorticity

- In the case $N_\varepsilon \to \infty$, describe the vortices via the **vorticity** :

  supercurrent

  $$j_\varepsilon := \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \qquad \langle a, b \rangle := \frac{1}{2}(a\bar{b} + \bar{a}b)$$

  vorticity

  $$\mu_\varepsilon := \mathrm{curl}\, j_\varepsilon$$

- $\simeq$ vorticity in fluids, but quantized: $\mu_\varepsilon \simeq 2\pi \sum_i d_i \delta_{a_i^\varepsilon}$
- $\frac{\mu_\varepsilon}{2\pi N_\varepsilon} \to \mu$ signed measure, or probability measure,

Dynamics in the case $N_\varepsilon \gg 1$

$$\frac{N_\varepsilon}{|\log \varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2 \qquad \text{(PGL)}$$

$$iN_\varepsilon \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2 \qquad \text{(GP)}$$

- For (GP), by Madelung transform, the limit dynamics is expected to be the 2D incompressible Euler equation. Vorticity form

$$\partial_t \mu - \operatorname{div}\,(\mu \nabla^\perp h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(EV)}$$

- For (PGL), formal model proposed by [Chapman-Rubinstein-Schatzman '96], [E '95]: if $\mu \geq 0$

$$\partial_t \mu - \operatorname{div}\,(\mu \nabla h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(CRSE)}$$

Studied by [Lin-Zhang '00, Du-Zhang '03, Masmoudi-Zhang '05, Ambrosio-S '08, S-Vazquez '13]

Dynamics in the case $N_\varepsilon \gg 1$

$$\boxed{\frac{N_\varepsilon}{|\log \varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2} \qquad \text{(PGL)}$$

$$\boxed{iN_\varepsilon \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2} \qquad \text{(GP)}$$

- For (GP), by Madelung transform, the limit dynamics is expected to be the 2D incompressible Euler equation. Vorticity form

$$\partial_t \mu - \mathrm{div}\,(\mu \nabla^\perp h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(EV)}$$

- For (PGL), formal model proposed by [Chapman-Rubinstein-Schatzman '96], [E '95]: if $\mu \geq 0$

$$\partial_t \mu - \mathrm{div}\,(\mu \nabla h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(CRSE)}$$

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Previous rigorous convergence results

- (PGL) case : [Kurzke-Spirn '14] convergence of $\mu_\varepsilon/(2\pi N_\varepsilon)$ to μ solving (CRSE) under assumption $N_\varepsilon \leq (\log\log|\log\varepsilon|)^{1/4}$ + well-preparedness
- (GP) case: [Jerrard-Spirn '15] convergence to μ solving (EV) under assumption $N_\varepsilon \leq (\log|\log\varepsilon|)^{1/2}$ + well-preparedness
- both proofs "push" the fixed N proof (taking limits in the evolution of the energy density) by making it more quantitative
- difficult to go beyond these dilute regimes without controlling distance between vortices, possible collisions, etc

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Modulated energy method

- Exploits the regularity and stability of the solution to the limit equation
- Works for dissipative as well as conservative equations
- Works for gauged model as well

Let $\mathrm{v}(t)$ be the expected limiting velocity field (such that $\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \rightharpoonup \mathrm{v}$ and $\mathrm{curl}\,\mathrm{v} = 2\pi\mu$). Define the modulated energy

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2},$$

modelled on the Ginzburg-Landau energy.

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Main result: Gross-Pitaevskii case

Theorem (S. '16)

Assume u_ε solves (GP) and let N_ε be such that $|\log \varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon}$. Let v be a $L^\infty(\mathbb{R}_+, C^{0,1})$ solution to the incompressible Euler equation

$$\begin{cases} \partial_t \mathrm{v} = 2\mathrm{v}^\perp \mathrm{curl}\, \mathrm{v} + \nabla \mathrm{p} & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases} \qquad \text{(IE)}$$

with $\mathrm{curl}\, \mathrm{v} \in L^\infty(L^1)$.
Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions associated to initial conditions u_ε^0, with $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \le o(N_\varepsilon^2)$. Then, for every $t \ge 0$, we have

$$\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, i u_\varepsilon \rangle \to \mathrm{v} \quad \text{in } L^1_{loc}(\mathbb{R}^2).$$

Implies of course the convergence of the vorticity $\mu_\varepsilon / N_\varepsilon \to \mathrm{curl}\, \mathrm{v}$
Works in 3D

Main result: parabolic case

Theorem (S. '16)

Assume u_ε *solves (PGL) and let* N_ε *be such that* $1 \ll N_\varepsilon \leq O(|\log \varepsilon|)$. *Let* v *be a* $L^\infty([0,T], C^{1,\gamma})$ *solution to*

- *if* $N_\varepsilon \ll |\log \varepsilon|$

$$\begin{cases} \partial_t \mathrm{v} = -2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} + \nabla \mathrm{p} & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{L1}$$

- *if* $N_\varepsilon \sim \lambda |\log \varepsilon|$

$$\partial_t \mathrm{v} = -2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} + \frac{1}{\lambda}\nabla \mathrm{div}\ \mathrm{v} \quad \text{in } \mathbb{R}^2. \tag{L2}$$

Assume $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \leq \pi N_\varepsilon |\log \varepsilon| + o(N_\varepsilon^2)$ *and* $\mathrm{curl}\,\mathrm{v}(0) \geq 0$. *Then* $\forall t \geq 0$ *we have*

$$\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, i u_\varepsilon \rangle \to \mathrm{v} \quad \text{in } L^1_{loc}(\mathbb{R}^2).$$

Taking the curl of the equation yields back the (CRSE) equation if $N_\varepsilon \ll |\log \varepsilon|$, but *not* if $N_\varepsilon \propto |\log \varepsilon|$!
Long time existence proven by [Duerinckx '16].

Proof method

- Go around the question of minimal vortex distances by using instead the modulated energy and showing a Gronwall inequality on $\mathcal{E}$.
- the proof relies on algebraic simplifications in computing $\frac{d}{dt}\mathcal{E}_\varepsilon(u_\varepsilon(t))$ which reveal only quadratic terms
- Uses the regularity of v to bound corresponding terms
- An insight is to think of v as a spatial gauge vector and $\mathrm{div}\ \mathrm{v}$ (resp. p) as a temporal gauge

THANK YOU FOR YOUR ATTENTION!