Lyapunov exponents

from the 1960s to the 2020s

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A few recent books

P. Duarte and S. Klein, *Continuity of the Lyapunov exponents of linear cocycles*, Publicações do IMPA, IMPA, 2017.

L. Barreira, *Lyapunov exponents*, Birkhäuser/Springer, Cham, 2017.

A. Pikovsky and A. Politi, *Lyapunov exponents. A tool to explore complex dynamics*, Cambridge University Press, 2016.

P. Duarte and S. Klein, *Lyapunov exponents of linear cocycles. Continuity via large deviations*, Atlantis Studies in Dynamical Systems, 3, Atlantis Press, 2016.

M. Viana, *Lectures on Lyapunov exponents*, Cambridge Studies in Advanced Mathematics, 145, Cambridge University Press, 2014.

N. Izobov, *Lyapunov exponents and stability*, Stability, Oscillations and Optimization of Systems, 6, Cambridge Scientific Publishers, 2012.

W. Siegert, *Local Lyapunov exponents. Sublimitting growth rates of linear random differential equations*, Lecture Notes in Mathematics, 1963, Springer-Verlag, 2009

Lyapunov stability

Consider the differential equation

$$x' = A(t)x + R(t,x), \quad R(t,0) \equiv 0. \tag{1}$$

The Lyapunov exponent function is defined by

$$\lambda(v) = \limsup_{t\to+\infty} \frac{1}{t} \log \|\Gamma(t)v\|,$$

where $t \mapsto \Gamma(t)$ is the fundamental solution of the linearized equation $x' = A(t)x$.

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Lyapunov stability theorem, 1892

$\lambda(v) < 0$ for every v together with "Lyapunov regularity" implies that the constant solution $x(t) \equiv 0$ is exponentially stable for equation (1).

Extremal Lyapunov exponents

Let ν be a probability measure on $\mathrm{GL}(d)$, such that $g \mapsto \log \|g^{\pm 1}\|$ are in $L^1(\nu)$.

Let $(g_n)_n$ be independent random variables in $\mathrm{GL}(d)$, all with probability distribution ν.

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Furstenberg–Kesten ergodic theorem, 1960

There exist numbers $\lambda_-(\nu)$ and $\lambda_+(\nu)$, called extremal Lyapunov exponents, such that

$$\lim_n \frac{1}{n} \log \|g_n \cdots g_1\| = \lambda_+(\nu) \text{ and}$$

$$\lim_n -\frac{1}{n} \log \|(g_n \cdots g_1)^{-1}\| = \lambda_-(\nu)$$

ν-almost surely. Moreover, $\lambda_-(\nu) \leq \lambda_+(\nu)$.

Linear cocycles

Let (M, μ) be a probability space and $f : M \to M$ be a measure-preserving map.

A linear cocycle over f is a map $F : \mathcal{V} \to \mathcal{V}$, where $\pi : \mathcal{V} \to M$ is a finite dimension vector bundle, such that the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{F} & \mathcal{V} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[f]{} & M \end{array}$$

commutes, and whose action $F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}$ on each fiber of $\mathcal{V}$ is linear.

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Example

$\mu = \nu^{\mathbb{Z}}$ a Bernoulli measure on $M = \mathrm{GL}(d)^{\mathbb{Z}}$, $f : M \to M$ the left-translation (shift), $\mathcal{V} = M \times \mathbb{R}^d$ a trivial vector bundle, and

$$F(\mathbf{g}, v) = (f(\mathbf{g}), g_0 v), \quad \text{where } \mathbf{g} = (g_n)_n.$$

Lyapunov exponents

Assume that the functions $x \mapsto \log \|F_x^{\pm 1}\|$ are in $L^1(\mu)$ and $f : M \to M$ is invertible.

Oseledets multiplicative ergodic theorem, 1968

For μ-almost every $x \in M$, there are numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, called Lyapunov exponents, and a splitting $\mathcal{V}_x = E_x^1 \oplus \cdots \oplus E_x^k$ such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|F_x^n v\| = \lambda_j(x) \text{ for } v \in E_x^j.$$

Moreover, $F_x(E_x^j) = E_{f(x)}^j$ and $\lambda_j(f(x)) = \lambda_j(x)$ at ν-almost every point.

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The heart of the proof is showing that μ-almost every orbit has Lyapunov regularity.

Non-uniform hyperbolicity

Let $f : M \to M$ be a diffeomorphism and $F = Df : TM \to TM$ be the derivative.

We call (f, μ) non-uniformly hyperbolic if the Lyapunov exponents of $F = Df$ are non-zero ν-almost everywhere.

Then there is a (measurable) hyperbolic dichotomy $T_x M = E_x^s \oplus E_x^u$, where

$$E_x^s = \bigoplus_{\lambda_j(x)<0} E_x^j \quad \text{and} \quad E_x^u = \bigoplus_{\lambda_j(x)>0} E_x^j.$$

Stable manifold theorem

Pesin stable manifold theorem, 1976

There is a measurable family of embedded smooth disks $W^s_{loc}(x)$ tangent to E^s_x at ν-almost every point and consisting of points that are forward-asymptotic to x.

Applying the theorem to the inverse f^{-1}, we get a corresponding statement for E^u_x.

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This is the starting point of one of the most fruitful areas of smooth dynamics, called Pesin (or non-uniform hyperbolicity) theory.

Question: How general is non-uniform hyperbolicity (almost everywhere non-vanishing Lyapunov exponents) among dynamical systems?

Partially hyperbolic dynamics

A diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a continuous decomposition

$$T_x M = E_x^u \oplus E_x^c \oplus E_x^s$$

(defined at **every** point) which is invariant under the dynamics:

$$Df_x(E_x^*) = E_{f(x)}^* \text{ for all } * \in \{u, c, s\}$$

and ...

Partially hyperbolic dynamics

- E^s is uniformly contracting:

$$\|Df_x \mid_{E^s_x}\| \leq \lambda < 1$$

- E^u is uniformly expanding:

$$\|(Df_x \mid_{E^u_x})^{-1}\| \leq \lambda < 1$$

- E^c is "in between":

$$\frac{1}{\lambda}\frac{\|Df_x(v^s)\|}{\|v^s\|} \leq \frac{\|Df_x(v^c)\|}{\|v^c\|} \leq \lambda \frac{\|Df_x(v^u)\|}{\|v^u\|}$$

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Question: How often are the center (i.e. along E^c) Lyapunov exponents non-vanishing?

Examples of partial hyperbolicity

Fact: Partial hyperbolicity is an open property.

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- Take $A \in \mathrm{SL}(d, \mathbb{Z})$ whose spectrum intersects the interior, the boundary, and the exterior of the unit disk in $\mathbb{C}$. Then the induced map is partially hyperbolic:

$$f_A : \mathbb{T}^d \to \mathbb{T}^d, \quad \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$$

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- Let $f^t : M \to M$, $t \in \mathbb{R}$ be an Anosov flow: there is an invariant decomposition

$$T_x M = E^u_x \oplus \mathbb{R} X(x) \oplus E^s_x, \quad X = \text{ associated vector field.}$$

Then the time–1 map f^1 is partially hyperbolic.

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- Let $g : N \to N$ be Anosov. Then any isometry extension is partially hyperbolic:

$$f : N \times \mathbb{T}^d \to N \times \mathbb{T}^d, \quad f(x, v) = \big(g(x), v + \omega(x)\big).$$

Smooth cocycles

Let (M, μ) be a probability space and $f : M \to M$ be a measure-preserving map.

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Example

The projectivization $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ of a linear cocycle $F : \mathcal{V} \to \mathcal{V}$: the fibers $\mathcal{E}_x = \mathbb{P}(\mathcal{V}_x)$ and each $\mathfrak{F}_x$ is the projectivization of the linear map F_x.

Fibered Lyapunov exponents

For $z \in \mathcal{E}$ and v a tangent vector to the fiber at z, the extremal Lyapunov exponents are

$$\lambda_+(z,v) = \lim_{n\to\infty} \frac{1}{n} \log \|D\mathfrak{F}_z^n(v)\| .$$

$$\lambda_-(z,v) = \lim_{n\to\infty} -\frac{1}{n} \log \|D\mathfrak{F}_z^n(v)^{-1}\| .$$

The limits exist m-almost everywhere if m is an $\mathfrak{F}$-invariant probability measure (Kingman subadditive ergodic theorem, 1968).

We are only interested in measures m that project down to μ under $\pi : \mathcal{E} \to M$.

Avila–Viana invariance principle, 2010

Let $\mathcal{A}$ be a generating σ-algebra in M such that both f and $x \mapsto \mathfrak{F}_x$ are $\mathcal{A}$-measurable. If $\lambda_-(z, v) \geq 0$ at m-almost every point then the disintegration $x \mapsto m_x$ of m along the fibers is $\mathcal{A}$-measurable.

Applying the theorem to the inverse, we get a dual statement when $\lambda_+(\mathfrak{F}, z, v) \leq 0$.

This extends results of Furstenberg, Ledrappier and Bonatti–Viana for linear cocycles.

Lyapunov exponents of partially hyperbolic maps

Let us go back to the partially hyperbolic setting:

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The center Lyapunov exponents of f are the numbers

$$\lambda(v^c) = \lim_n \frac{1}{n} \log \|Df_x^n(v^c)\| \text{ of vectors } v^c \in E_x^c$$

They are well defined almost everywhere (Oseledets theorem).

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They are well defined almost everywhere (Oseledets theorem).

Question: Can we always perturb f to make the center Lyapunov exponents non-zero?

Symplectic diffeomorphisms

Let $f_A : \mathbb{T}^4 \to \mathbb{T}^4$ be induced by some linear map $A \in \mathrm{SL}(4, \mathbb{Z})$ with exactly two eigenvalues in the unit circle.

Basic facts:

- f_A preserves some (constant) symplectic form ω.
- f_A preserves volume.
- Assuming that no eigenvalue is a root of unit, f_A is ergodic.

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F. Rodriguez-Hertz stable ergodicity theorem, 2005

Every volume-preserving diffeomorphism f close to f_A is ergodic.

Symplectic diffeomorphisms

In fact, f is stably Bernoulli among symplectic diffeomorphisms:

Avila, Viana stable Bernoulli theorem, 2010

Let $f : \mathbb{T}^4 \to \mathbb{T}^4$ be any ω-symplectic diffeomorphism close to f_A. Then:

- either f has all center Lyapunov exponents non-zero,
- or f is conjugate to f_A by a volume-preserving diffeomorphism.

In either case, f is ergodically equivalent to a Bernoulli shift.

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The proof involves several applications of the invariance principle.

Direct perturbation of Lyapunov exponents

A different application of the invariance principle yields a direct proof that vanishing exponents can be disposed of, at least in some cases:

Let $f : M \to M$ be a partially hyperbolic, symplectic, C^k diffeomorphism having some periodic point.

Marín, 2016

Assume that f is accessible, center-bunched and pinched and the center bundle E^c is 2-dimensional. Then f is C^k-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.

Many other important issues

- simplicity of the Lyapunov spectrum
- dependence of the Lyapunov exponents on the cocycle
- Schrödinger cocycles, random or quasi-periodic
- dynamics of group actions
- numerical analysis of Lyapunov exponents
- ...

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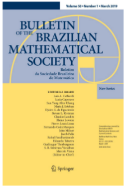

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