To bem, or not to bem, that is the question

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1) Joint work with Sebastian Schöps and Felix Wolf
Outline

1. Introduction
2. BEM in a Nutshell
3. Advantages and Challenges
4. Isogeometric Analysis
5. Summary and Outlook
## Popular Numerical Methods

<table>
<thead>
<tr>
<th>Domain type</th>
<th>Equation type</th>
<th>Domain oriented</th>
<th>Boundary oriented</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Partial differential equation, e.g. $\Delta u = 0$</td>
<td>Finite Differences, ~ Volumes, ~ Element Method, ~ Integration Technique</td>
<td>Discretization of shells</td>
</tr>
<tr>
<td></td>
<td>Integral equation, e.g. $u(x) = \int_{\Gamma} \frac{w(y)}{</td>
<td>x - y</td>
<td>} , d\Gamma_y$</td>
</tr>
</tbody>
</table>
BEM in a Nutshell

The term Boundary Element Method (BEM) denotes any method for the approximate numerical solution of boundary integral equations. [Costabel 1987]
Electrostatic Problem (i)

\[ u = ? \]

\[ \Omega \]

\[ \Delta u = 0 \]

Mathematical model

Representation formula

\[ u(x) = \int_{\Gamma} \frac{w(y)}{4\pi|x-y|} \, d\Gamma_y \]

Single layer potential

\[ \Gamma : u \overset{!}{=} g \]

\[ w = ? \]
Electrostatic Problem (ii)

\[ x \to \Gamma : u \to g \]

\[ g(x) = \int_{\Gamma} \frac{w(y)}{4\pi|x - y|} \, d\Gamma_y \quad \{ \text{Fredholm 1st kind} \} \]

Find \( w(y) \in \mathcal{X}(\Gamma) \) such that \( \forall w'(x) \in \mathcal{X}(\Gamma) \)

\[
\int_{\Gamma} g(x)w'(x) \, d\Gamma_x = \int_{\Gamma} \int_{\Gamma} \frac{w(y)w'(x)}{4\pi|x - y|} \, d\Gamma_y \, d\Gamma_x
\]

Single layer operator, fundamental solution
Electrostatic Problem (iii)

Mathematical model

Representation formula

Boundary integral equation

Weak formulation

Boundary Elements

\[ w \approx w_\ell \]

piecewise constant
Electrostatic Problem (iv)

\[ v_{k,\ell} := \int_{\tau_k} \int_{\tau_{\ell}} \frac{1}{4\pi |x - y|} \, d\Gamma_y d\Gamma_x \]

\[ f_k := \int_{\tau_k} g(x) \, d\Gamma_x \]

\[ \mathbf{Vw} = \mathbf{f} \quad \text{Galerkin discretization} \]

\[ \int_{\Gamma} \frac{w(y)}{4\pi |x - y|} \, d\Gamma_y = u(x) \quad \checkmark \]
### Application Domains

<table>
<thead>
<tr>
<th>Equation</th>
<th>Application domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace equation</td>
<td>Electric, magnetic, gravitational, fluid potentials; steady-state heat conduction</td>
</tr>
<tr>
<td>Helmholtz equation</td>
<td>Acoustic radiation and scattering</td>
</tr>
<tr>
<td>Maxwell equations</td>
<td>Electromagnetic radiation and scattering</td>
</tr>
<tr>
<td>Lamé equations</td>
<td>Linear isotropic elastostatics</td>
</tr>
<tr>
<td>Stokes equations</td>
<td>Incompressible flows at low Reynolds number (e.g. around red blood cells)</td>
</tr>
</tbody>
</table>

Credit: Y. Liu, [Link](#)
Coupling of Boundary and Finite Elements: Design of the Large Hadron Collider @ CERN

2D

Yoke: FEM

Coupling interface

3D

Coil & collar: BEM

Currents: Biot-Savart

Coupling interface


Credit: CERN
Coupling of Boundary and Finite Elements: Design of Electrical Traction Drives @ Bosch

- Credit: Bosch
- Credit: S. Schöps PASIROM

Diagram showing:
- Stator: FEM
- Rotor: FEM
- Air gap: BEM

Graph with:
- OP₁ and OP₂
- Power
- Torque
- η ≥ 92%
Advantages and Challenges
We discuss matrix compression techniques and preconditioning
## Advantages and Challenges

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Challenges</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Only boundary</strong> of domain needs to be discretized, fits B-REP of CAD systems</td>
<td>Restricted to linear PDE(^1), (mostly) with constant coefficients</td>
</tr>
<tr>
<td><strong>Exterior</strong> problems handled as easily as <strong>interior</strong> problems</td>
<td>Mathematics more involved</td>
</tr>
<tr>
<td>Rather <strong>high convergence rate in interior</strong> of <strong>domain</strong>, holds for derivatives of any order</td>
<td>Naive implementation yields <strong>fully populated matrices</strong>: often, compression schemes are required (ACA(^2), FMM(^3))</td>
</tr>
</tbody>
</table>

\(^1\) Partial Differential Equation  \(^2\) Adaptive Cross Approximation  \(^3\) Fast Multipole Method
Optimization of a Claw-Pole Alternator
Adaptive Cross-Approximation (ACA)

Air gap: block-circulant BEM matrix, due to symmetry

\[
V = \begin{pmatrix}
V^1 & V^2 & \cdots & V^m \\
V^m & V^1 & \cdots & V^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
V^2 & V^3 & \cdots & V^1
\end{pmatrix}
\]

~84.000 DOF in total, \( m = 12 \)

ACA compression rates

Scattering at a Bent Graded-Index Fiber
Calderón Preconditioner: \(4DV = I + \text{compact}\)

Exterior: BEM, \(\varepsilon = \varepsilon_0\)

Fiber: FEM, \(\varepsilon = \varepsilon(r)\)

D. Dobbelare et al.: A Calderón multiplicative preconditioner for the electromagnetic Poincaré–Steklov operator..., 2015

S.P. Adrian, F.P. Andriulli & T.F. Eibert: On a refinement-free Calderón multiplicative preconditioner for the EFIE, 2019

TFQMR iterations until \(\text{err} = 10^{-6}\)

- no precon: \(2223\)
- Jacobi precon: \(408\)
- Calderón precon: \(46\)
Isogeometric Analysis

Co-evolution of Computer-Aided Design and Finite-Element Analysis

J. Dölz, S. Kurz, S. Schöps, F. Wolf: An Overview of Isogeometric Boundary Element Methods for Acoustic and Electromagnetic Scattering Problems, 2018

- : Isogeometric Boundary Elements in Electromagnetism: Rigorous Analysis, Fast Methods, and Examples, arXiv preprint 2018

- : A Numerical Comparison of an Isogeometric and a Classical Higher-Order Approach to the Electric Field Integral Equation, arXiv preprint 2018

Synchronism of particles and field is fundamental
REQ: Accuracy of $10^{-6}$ of the field approximation along the axis
REQ: Exact geometry representation
A Blend of Discretization Methods

- Boundary representation from CAD by NURBS patches
- Conic sections can be exactly represented
- No further discretization, no volume segmentation
- Displacements due to electromagnetic forces can be directly applied (Lorentz detuning)
- Design optimization on the geometry level

Computer-Aided Design CAD

Finite Element Method FEM

linear, homogeneous

Boundary Element Method BEM

BEM & Isogeometric Analysis IGA-BEM

TESLA Cell
IGA-BEM: Electric Field Integral Equation

Find eigenvalue $\kappa \in \mathbb{R}^+$ and surface current $\mathbf{w}(y) \in \mathcal{X}(\Gamma)$, $\mathbf{w}(y) \neq 0$, such that $\forall \mathbf{w}'(x) \in \mathcal{X}(\Gamma)$

$$\int_{\Gamma} \int_{\Gamma} \left( \mathbf{w} \cdot \mathbf{w}' - \frac{1}{\kappa^2} \text{div}_\Gamma \mathbf{w} \text{div}_\Gamma \mathbf{w}' \right) u^*_\kappa \, d\Gamma_y d\Gamma_x = 0$$

Fundamental solution $u^*_\kappa(r) = \frac{\exp(-i\kappa r)}{4\pi r}$

$r = |x - y|$

Holomorphic operator $V : \mathcal{D} \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{X}')$

$V(\kappa) \mathbf{w} = 0$, $\kappa \in \mathcal{D} \subset \mathbb{C}$

IGA-BEM: Definition of B-Splines

- Define B-Splines recursively
  - piecewise constant $\rightarrow$ linear $\rightarrow$ ...
  - polynomial degree $p$
  - knot vector $\Xi$
  - dimension of spline space $k$

- NURBS from B-Splines by weighting & normalization

- B-Splines and NURBS admit simple tensor product constructions

- A discrete de Rham complex can be built from B-Spline spaces

\[ p = 1, \quad k = 4 \]
\[ \Xi = [0; 0; \frac{1}{3}; \frac{2}{3}; 1; 1] \]

\[ p = 2, \quad k = 5 \]
\[ \Xi = [0; 0; 0; \frac{1}{3}; \frac{2}{3}; 1; 1; 1] \]
## IGA-BEM: Discrete de Rham Spline Complex

<table>
<thead>
<tr>
<th>Surface quantity</th>
<th>Potential</th>
<th>Current</th>
<th>Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patch gluing</td>
<td>Continuous</td>
<td>Normal continuous</td>
<td>Discontinuous</td>
</tr>
</tbody>
</table>

- $\mathcal{S}^0 \subset H^{1/2}$
- $\mathcal{S}^1 \subset \mathcal{X} = H_{\times}^{-1/2} (\text{div}_\Gamma)$
- $\mathcal{S}^2 \subset H^{-1/2}$
Comparison to Raviart-Thomas (RT) Elements

**Toy boat example**
- Scattering from a dipole
- 28 patches
- Interpolation-based fast multipole method
- Complex GMRES, no preconditioner
- Field evaluation from surface currents

**Max. pointwise error**
- Theoretical convergence $O(h^{2p+1})$

**Accuracy per DOF**
- for spline spaces

**Size of resulting discrete system**

- $B$-Spline
- RT

- $p = 2$
- $p = 3$

- $O(h^7)$
- $O(h^5)$
Summary and Outlook

Release of BEMBEL

Boundary Element Method Based Engineering Library successfully released under GNU GPLv3 03.2019 in cooperation with Universities of Basel and Lugano

www.bembel.eu
To BEM, or not to BEM, that is the question

Summary and Outlook

- BEM has evolved into an important alternative to domain-oriented methods
  - Ever since “fast” methods available
  - Exterior radiation and scattering problems
  - Coupling of boundary and finite elements
  - Perfect match with Isogeometric Analysis

- What’s next?
  - BEM in space-time, with O. Steinbach, TU Graz
  - In space-time, motion is geometry

S. Dohr et al.: *A parallel space–time boundary element method for the heat equation*, in press